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Best Approximations in Locally Convex Spaces*

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The results of Brosowski, Meinardus, and Singh are extended and unified.

1. INTRODUCTION

Applications of fixed point theorems to approximation theory are well known. Using the fixed point theory there are very elegant proofs of results such as "a boundedly compact Chebyshev set is a sun" and "every boundedly compact Chebyshev set in a smooth Banach space is convex." Brosowski [1], Meinardus [5], and Singh [7, 8] used the fixed point theory to prove some other interesting theorems in approximation theory.

Singh [7] stated in the introduction that the mapping considered is continuous, not necessarily Lipschitzian. We would like to point out that the continuity and condition $p(Tx - Tb) \leqslant p(x - b)$ for all x in D are superfluous in [8, Theorem 1], as they are a direct consequence of nonexpansiveness.

Throughout this paper E will denote a Hausdorff locally convex linear topological space "T_2 *lts*" and Q a (fixed) family of continuous seminorms

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which generates the topology of E. Let C be a nonempty subset of E and p be a continuous seminorm on E. For $x \in E$ define

$$d_p(x, C) = \inf\{p(x-y): y \in C\}$$

and

$$R_p(x) = \{x \in C: p(x-y) = d_p(x, C)\}.$$

The set C is said to be *proximinal* with respect to p if for all x in C, $R_p(x)$ is nonempty [6, p. 105].

DEFINITION 1.1. A mapping $T: C \to C$ is said to be a *p-contraction* if there is a k_p, $0 \leqslant k_p < 1$ such that

$$p(Tx - Ty) \leqslant k_p\, p(x-y)$$

for all $x, y \in C$ and $p \in Q$.

The mapping, T is called *p-nonexpansive* if for all $x, y \in C$ and $p \in Q$

$$p(Tx - Ty) \leqslant p(x-y).$$

DEFINITION 1.2. A mapping $T: C \to C$ is said to be *demicompact* if each bounded net $\{x_n\}$ in C such that $(I-T)(x_n)$ converges has a convergent subnet.

DEFINITION 1.3. A subset K of E is *star-shaped* provided there is a point $p \in K$ such that for each $x \in K$, the segment joining x to p is contained in K (i.e., there is $p \in K$ such that $tx + (1-t)p \in K$ for each $x \in K$ and real t with $0 < t < 1$). Such a point p will be called a *star centre* of K.

Clearly, if K is convex, then every point in K is a star centre of K.

DEFINITION 1.4. Let $\{x_n\}$ be a sequence in E. Then $\{x_n\}$ is *Cauchy* if and only if for each $p \in Q$, $p(x_n - x_m) \to 0$ as $n, m \to \infty$. Space E is *sequentially complete* if and only if every Cauchy sequence in E converges to some element in E. Space E is *quasicomplete* if every bounded, closed subset of E is complete.

Clearly every complete space is quasicomplete, and every quasicomplete space is sequentially complete [4].

2. Main Result

THEOREM 2.1. *Let E be a T_2 lts, and $T: E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant set and y a T-invariant point. Assume that for every $p \in Q$ the set D of Best C-approximants to y with respect to p is nonempty, sequentially complete, bounded, and star-shaped. Furthermore, assume either of the following holds:*

(i) *$(I - T(D)$ is closed,*

(ii) *T is demicompact.*

Then T has a fixed point which is a Best approximation to y in D.

Proof. Since D is the set of Best C-approximants to y, it follows that T maps D into itself. Indeed, if $x \in D$, then using the p-nonexpansiveness of T we have

$$p(Tx - y) = p(Tx - Ty) \leqslant p(x - y),$$

implying that $T(x)$ is in D. Let q be the star centre of D. For each t, $0 < t < 1$, define

$$T_t(x) = tT(x) + (1 - t)q \qquad (x \in D).$$

Since D is star-shaped, each T_t clearly maps D into itself. Also T_t is a contraction; in fact

$$\begin{aligned} p(T_t(x) - T_t(y)) &= p(tT(x) - tT(y)) \\ &\leqslant tp(Tx - Ty) \leqslant tp(x - y) \end{aligned}$$

for all x, y in D. Since D is complete it follows that T_t has a unique fixed point x_t in D [2, Theorem 2.2]. Now

$$\begin{aligned} (I - T)(x_t) = x_t - Tx_t &= x_t - (1/t)(T_t x_t - (1 - t)q) \\ &= (1 - (1/t))(x_t - q), \end{aligned}$$

which clearly tends to zero in E as $t \to 1$, since D is bounded. Since $(I - T)(D)$ is closed, there is a $x \in D$ such that $x - Tx = 0$, so x is a fixed point of T.

(ii) We show that $(I - T)(D)$ is closed and the result will follow from (i). Let $\{(I - T)x_\alpha : \alpha \in \Delta\}$ be a net in D such that $(I - T)x_\alpha \to y$. Since T is demicompact, $\{x_\alpha\}$ contains a convergent subnet which we denote by $\{x_{\alpha'}\}$. Since D is closed we have $x_{\alpha'} \to x$, where x is some point of D. By the continuity of T it follows that $(I - T)(x_{\alpha'}) \to (I - T)x$. Then $(I - T)x = y$.

3. Corollaries

Corollary 3.1 [8, Theorem 1]. *Let E be a T_2 lts and $T: E \to E$ be p-nonexpansive. Let C be a T-invariant subset of E and y a T-invariant point. If the set of Best C-approximants to y is nonempty, compact, and star-shaped, then it contains a T-invariant point.*

Proof. Let D be the set of Best C-approximants to y. Then being compact it is complete and totally bounded, and hence sequentially complete and bounded. Moreover, it follows from the continuity of $I - T$ that $(I - T)(D)$ is compact, and hence closed. Thus the corollary follows.

Corollary 3.2. *Let E be a T_2 lts and $T: E \to E$ be nonexpansive. Let C be a T-invariant subset of E and y be a T-invariant point. If the set of Best C-approximants to y is nonempty, quasicomplete, bounded, and star-shaped, then it contains a T-invariant point, provided T is demicompact.*

Corollary 3.3. *Let X be a normed linear space and $T: X \to X$ be a nonexpansive mapping. Let T have a fixed point, say y, and a T-invariant finite-dimensional subspace C of X. Then T has a fixed point which is a Best approximation to y in C.*

Proof. It is well known that the set D of Best C-approximants to y is nonempty. Also D is closed, bounded, and convex. Since C is finite-dimensional, D is compact and hence the result follows from the corollary.

An immediate consequence of Corollary 3.1. is

Corollary 3.4 [7]. *Let T be a nonexpansive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of Best C-approximants to x is nonempty, compact, and star-shaped, then it contains a T-invariant point.*

The following well-known result of Meinardus [5] follows from Corollary 3.3:

Corollary 3.5 [5, Theorem 27]. *Let B be a compact metric space and $T: B \to B$ continuous. Let $C[B]$ be the space of all continuous real or complex functions on B with the sup norm. Let $A: C[B] \to C[B]$ be Lipschitzian with Lipschitz constant 1. Suppose further that $Af(T(x)) = f(x)$, $Ah(Tx) \in V$ whenever $h(x) \in V$, where V is a finite-dimensional subspace of $C[B]$. Then there is a Best approximation g of f with respect to V such that*

$$Ag(Tx) = g(x).$$

Proof. Define the mapping S by

$$Sg = Ag(Tx).$$

Then S is nonexpansive. Indeed using the linearity of A we have

$$\|Sg - Sg_1\| = \|Ag(Tx) - Ag_1(Tx)\|$$
$$\leqslant \|g(Tx) - g_1 T(x)\| \leqslant \|g - g_1\|.$$

Now the set of Best approximations is convex and closed. An appeal to the Arzela–Ascoli Theorem guarantees the compactness, hence the corollary.

Before we state our next result we need

DEFINITION 3.1. Let C be a subset of E and $T: C \to E$ be a mapping. The mapping T is said to be *demiclosed* if for any net x_α in C such that $x_\alpha \to x$ weakly and $Tx_\alpha \to y$ it follows that $y = Tx$. In other words, the mapping T is demiclosed if its graph in $C \times E$ is closed in the Cartesian product topology induced in $C \times E$ by the weak topology in C and the strong topology in E, the mapping T is *weakly pseudocontinuous* if and only if for each neighbourhood U of zero there is a neighbourhood V of zero, $V \subset U$ such that

$$g^{-1}(\bar{V}) = \{x \in C: g(x) \in \bar{V}\}$$

is a weakly closed subset of V.

COROLLARY 3.6. *Let E be a T_2 lts and $T: E \to E$ be a p-nonexpansive mapping. Let C be a T-invariant set and y a T-invariant point. Let the set D of Best C-approximants to y with respect to p be nonempty, weakly compact, and star-shaped. Suppose either of the following conditions are satisfied*:

(i) *$I - T$ is demiclosed,*

(ii) *$I - T$ is weakly pseudocontinuous.*

Then T has a fixed point which is a Best approximation to $y \in X$.

Proof. Since weakly compact subsets of a locally convex linear topological space are both bounded and complete [3, pp. 159–160], it follows from the proof of Theorem 2.1 that there is a net $\{x_\alpha\}$ in D such that $x_\alpha - T(x_\alpha) = (I - T)(x_\alpha) \to 0$. Since D is weakly compact, we may assume (by passing to a subnet if necessary) that $x_\alpha \to x$ weakly for some x in D. Since $(I - T)$ is demiclosed, it follows that $(I - T)(x) = 0$, i.e., x is a fixed point of T. Let B be the basis for U, where U is the family of neighbourhoods of zero for a linear topological space E. Let $W \in B$ and pick a neighbourhood $V \subset W$ satisfying the definition of pseudoweakly continuous. Since $\{(I-T)(x_\alpha)\}$ is residual in $\bar{V}$, we see that $\{x_\alpha\}$ is residual in $(I-T)^{-1}(\bar{V})$, which is weakly closed by hypothesis. Hence the weak limit x lies in $(I-T)^{-1}(\bar{V})$ and

$$(I-T)(x) = x - T(x) \in \bar{V} \subset \bar{W}$$

for all $W \in B$. This insures that $x - Tx = 0$, and x is a fixed point of T.

COROLLARY 3.7. *Let X be a Banach space and $T: X \to X$ be a nonexpansive mapping. Let C be a T-invariant and y a T-invariant point. Let D, the set of Best C-approximants to y be nonempty, weakly compact, and star-shaped. Furthermore, suppose $I-T$ is demiclosed; then T has a fixed point which is a Best approximation to y in C.*

Remark 3.1. If T is weakly continuous, then $I-T$ is clearly demiclosed. In particular, if T is linear (in fact even affine), then the continuity of T (a consequence of nonexpansiveness) implies T is weakly continuous.

Using Remark 3.1 we have the following result of [1]:

COROLLARY 3.8. *Let T be a nonexpansive linear operator on a normed linear space X. Let Y be a T-invariant subset of X and x a T-invariant point. If the set of Best Y-approximants to x is nonvoid, convex, and compact, then it contains a T-invariant point.*

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